Short solution of the radioactive decay chain equations

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In most textbooks on nuclear physics[1–3] the set of radioactive decay chain equations is solved analytically for a maximum of three nuclides. The general Bateman solution[4] is given as a final result or only with a brief mention of the elaborate recursive procedure needed to obtain it.[3] Here, a short method for obtaining the general solution is demonstrated. © *2002 American Association of Physics Teachers.*

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The set of equations for a nonbranched decay chain is

$$\begin{aligned}\frac{dN_1}{dt}&=-\lambda_1 N_1,\\ &\vdots\\ \frac{dN_k}{dt}&=\lambda_{k-1}N_{k-1}-\lambda_k N_k,\end{aligned}\tag{1}$$

where λ_k is the decay constant of the kth nuclide, and $N_k=N_k(t)$ is the number of kth nuclei at time t. We first assume simple initial conditions, for example, $N_1(t=0)=N_{10}$, $N_k(t=0)=0$ for $k\geqslant 2$. The Laplace transformation[5] is applied to both sides of Eq. (1):

$$\begin{aligned}\int_0^\infty e^{-st}\frac{dN_1}{dt}dt&=\int_0^\infty e^{-st}dN_1\\ &=e^{-st}N_1(t)\big|_0^\infty+s\int_0^\infty e^{-st}N_1(t)dt\\ &=-N_{10}+s\tilde{N}_1(s)=-\lambda_1\tilde{N}_1(s),\end{aligned}\tag{2a}$$

$$\begin{aligned}\int_0^\infty e^{-st}\frac{dN_k}{dt}dt&=e^{-st}N_k(t)\big|_0^\infty+s\int_0^\infty e^{-st}N_k(t)dt\\ &=s\tilde{N}_k(s)=\lambda_{k-1}\tilde{N}_{k-1}(s)-\lambda_k\tilde{N}_k(s),\end{aligned}\tag{2b}$$

where $\tilde{N}_k(s)$ is the Laplace transform of $N_k(t)$.

Equations (2a) and (2b) are algebraic in $\tilde{N}_k(s)$, and we find

$$\tilde{N}_1(s)=\frac{N_{10}}{\lambda_1+s},\tag{3a}$$

$$\begin{aligned}\tilde{N}_k(s)&=\frac{\lambda_{k-1}}{\lambda_k+s}\tilde{N}_{k-1}(s)\\ &=\cdots=\frac{\lambda_{k-1}\lambda_{k-2}\cdots\lambda_1}{(\lambda_k+s)(\lambda_{k-1}+s)\cdots(\lambda_1+s)}N_{10}.\end{aligned}\tag{3b}$$

The inverse Laplace transform of Eq. (3) is given by the Bromwich integral:[5]

$$N_k(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\tilde{N}_k(s)ds,\tag{4}$$

where the constant γ is chosen so that all singularities of $\tilde{N}_k(s)$ are on the left-hand side. In the present case all singularities at $(-\lambda_1,-\lambda_2,\ldots)$ are negative and simple poles. The integral may be closed by an infinite semicircle in the left half-plane of the complex plane. Because all $\tilde{N}_k(s)$ are of the form $P/Q(s)$, where P are constants and $Q(s)$ are polynomials of degree $\geqslant 1$, the path integral along the semicircle tends to zero as the radius tends to infinity.[5] Then by the residue theorem, we obtain

$$\begin{aligned}N_k(t)&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\tilde{N}_k(s)ds\\ &=\frac{1}{2\pi i}\oint_C e^{st}\tilde{N}_k(s)ds=\sum_{j=1}^{k}\mathrm{Res}_j[e^{st}\tilde{N}_k(s)],\end{aligned}\tag{5}$$

where Res_j denotes the residue of the function in the square brackets at $s=-\lambda_j$. Hence,

$$\begin{aligned}\mathrm{Res}_j[e^{st}\tilde{N}_k(s)]&=\lim_{s\to-\lambda_j}[(s+\lambda_j)e^{st}\tilde{N}_k(s)]\\ &=e^{-\lambda_j t}\frac{\lambda_{k-1}\ldots\lambda_1}{(\lambda_k-\lambda_j)\ldots(\lambda_1-\lambda_j)}N_{10}.\end{aligned}\tag{6}$$

Finally, by doing the sum, we obtain:

$$N_k(t)=N_{10}\lambda_{k-1}\lambda_{k-2}\ldots\lambda_1\sum_{j=1}^{k}\frac{e^{-\lambda_j t}}{\Pi_{i=1(i\neq j)}^{k}(\lambda_i-\lambda_j)}.\tag{7}$$

Alternatively, the same result can be obtained without complex integrals. According to the Heaviside theorem:[6]

$$N_k(t)=\sum_{j=1}^{k}\frac{P}{Q'(-\lambda_j)}e^{-\lambda_j t},\tag{8}$$

and the substitution of P and Q' leads to Eq. (7).

Equation (7) represents the Bateman solution for the given initial conditions. It is easily generalized to the most common case of nonzero initial conditions with $N_k>0$ for $k>0$. For example, if $N_m(t=0)=N_{m0}\neq 0$, we should add a similar expression for the subchain starting at $i>m$:

$$\begin{aligned}N_i(t)&=N_{10}\lambda_{i-1}\lambda_{i-2}\ldots\lambda_1\sum_{k=1}^{i}\frac{e^{-\lambda_k t}}{\Pi_{j=1(j\neq k)}^{i}(\lambda_j-\lambda_k)}\\ &+N_{m0}\lambda_{i-1}\ldots\lambda_m\sum_{k=m}^{i}\frac{e^{-\lambda_k t}}{\Pi_{j=m(j\neq k)}^{k}(\lambda_j-\lambda_k)}.\end{aligned}\tag{9}$$

In this way, for the most common case, $N_i(t=0)=N_{i0}$ forsome or all of the nuclides, the final general solution is obtained:

$$N_i(t)=N_{i0}\,e^{-\lambda_i t} + \sum_{m=1}^{i-1} N_{m0} \prod_{q=m}^{i-1} \lambda_q \sum_{k=m}^{i} \frac{e^{-\lambda_k t}}{\Pi_{j=m(j\neq k)}^{i}(\lambda_j-\lambda_k)}. \tag{10}$$

A generalization of this method for a branched decay chain is easy, but this case is normally outside the scope of graduate courses in nuclear physics.

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