Beyond the Standard Model

Lecture 4

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SUPERSYMMETRY

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SUSY

We are looking for non trivial unification of internal and space-time symmetries

If P is the Space-time group of symmetry

$$[P_a, P_b] = 0 \qquad [P_a, J_{bc}] = (\eta_{ab} P_c - \eta_{ac} P_b)$$

$$[J_{ab}, J_{cd}] = -(\eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad})$$

And G is the internal symmetry group

$$[T_r, T_s] = f_{rst} T_t$$

Coulmen - Mandela No-go theorem

If P and G are Li groups, it is not possible to find group SG, for which G $\subset$ SG and P $\subset$ SG, different from G $\otimes$ P

i.e. trivial unification

The way out – use something which is not a Li group ?!

Haag – Lopushanski - Sohnius

SUSY

The solution is a new kind of symmetry – between fermions and bosons

Group → Super group – new spinor generators

Super Algebra – commuting and anticommuting generators

$Q_\alpha|F\rangle \rightarrow |B\rangle$ and $Q_\alpha|B\rangle \rightarrow |F\rangle$

SM particles form supermultiplets

Equal number fermions and bosons – new particles

However particles in one multiplet have equal masses

Supersymmetry should be broken

SUSY SM

spin 0	spin 1/2	spin 1	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$\tilde{u}_L, \tilde{d}_L$	u_L, d_L		$\mathbf{3}$	$\mathbf{2}$	$+\frac{1}{3}$
$\tilde{u}_R$	u_R		$\mathbf{3}$	$\mathbf{1}$	$+\frac{4}{3}$
$\tilde{d}_R$	d_R		$\mathbf{3}$	$\mathbf{1}$	$-\frac{2}{3}$
$\tilde{\nu}, \tilde{e}_L$	ν, e_L		$\mathbf{1}$	$\mathbf{2}$	-1
$\tilde{e}_R$	e_R		$\mathbf{1}$	$\mathbf{1}$	-2
H_u^+, H_u^0	$\tilde{h}_u^+, \tilde{h}_u^0$		$\mathbf{1}$	$\mathbf{2}$	$+1$
H_d^0, H_d^-	$\tilde{h}_d^0, \tilde{h}_d^-$		$\mathbf{1}$	$\mathbf{2}$	-1
	$\tilde{g}$	g	$\mathbf{8}$	$\mathbf{1}$	0
	$\tilde{w}^\pm, \tilde{w}^0$	$W^\pm, W^0$	$\mathbf{1}$	$\mathbf{3}$	0
	$\tilde{b}^0$	B^0	$\mathbf{1}$	$\mathbf{1}$	0

Dimensions

Lets consider the dimension of the variables and fields:

[x] = m^{-1} [dx] = m^{-1} $[\partial_\mu] = m$

Action should be dimensionless i.e. $S = \int d^4 x L(\phi, \partial_\mu \phi)$

From [S] = 0 ➔ $[L(\phi, \partial_\mu \phi)] = m^4$

Scalar field: from $L \sim \partial_\mu \phi \partial^\mu \phi$ ➔ $[\phi] = m$

Spinor field: from $L \sim \overline{\psi} \partial^\mu \gamma_\mu \psi$ ➔ $[\psi] = m^{3/2}$

Vector field: same like scalar one $[A_\mu] = m$

Supermultiplets

Let us consider the scalar field first, and tentatively define

$$\delta\phi = \bar{\eta}\,\psi \qquad \delta\phi^* = \bar{\psi}\,\eta$$

where η is a constant, infinitesimal, anticommuting spinor. Then

$$\delta\mathcal{L}_s = (\partial^\mu\bar{\psi})\,(\partial_\mu\phi)\,\eta + \bar{\eta}\,(\partial^\mu\phi^*)\,(\partial_\mu\psi)$$

How does ψ tranform? $\delta\psi$ must be left-handed, linear in η, and must contain one derivative. Only one possibility (up to a proportionality factor):

$$\delta\psi = -i(\partial_\mu\phi)\,\gamma^\mu\eta \qquad \delta\bar{\psi} = i\bar{\eta}\,\gamma^\mu\partial_\mu\phi^*$$

which tells us that η is right-handed, $P_R\eta = \eta$, and gives

$$\delta\mathcal{L}_f = -\bar{\eta}\,(\partial_\mu\phi^*)\,\gamma^\mu\gamma^\nu\partial_\nu\,\psi + \bar{\psi}\,\gamma^\nu\gamma^\mu\partial_\nu\,(\partial_\mu\phi)\,\eta$$

Supermultiplets

Using $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ we get

$$\begin{aligned}\delta\mathcal{L}_f \quad = \quad & -\bar\eta\,(\partial_\mu\phi^*)\,(\partial^\mu\psi) - (\partial_\mu\bar\psi)\,(\partial^\mu\phi)\,\eta \\ & -\bar\eta\,\partial_\nu[(\partial_\mu\phi^*)\,\gamma^\mu\gamma^\nu\,\psi] + \bar\eta\,\partial_\mu[(\partial^\mu\phi^*)\,\psi] + \partial_\mu(\bar\psi\,\partial^\mu\phi)\,\eta\end{aligned}$$

The first row cancels against $\delta\mathcal{L}_s$, and the rest is a total derivative. So,

$$\delta \int d^4x\,\mathcal{L}_{\rm chiral} = 0$$

Not yet enough to declare that $\mathcal{L}_{\rm chiral}$ is supersymmetric: we must check that the commutator of two transformations is a symmetry of the theory. For the scalar fields we find

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})\phi = -(\bar\eta_2\,\gamma^\mu\,\eta_1 - \bar\eta_1\,\gamma^\mu\,\eta_2)\,i\partial_\mu\phi$$

Good news: $i\partial_\mu$ is nothing but the four-momentum operator P_μ, the generator of space-time translations, a symmetry of space-time.

Supermultiplets

Now consider the fermion fields. After some tedious algebra,* we get

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})\psi = -(\bar{\eta}_2\gamma^\mu\eta_1 - \bar{\eta}_1\gamma^\mu\eta_2)\, i\partial_\mu\psi + \frac{1}{2}(\bar{\eta}_2\gamma^\mu\eta_1 - \bar{\eta}_1\gamma^\mu\eta_2)\, i\gamma_\mu \not{\partial}\psi$$

The first term is similar to what we got for ϕ, while the second one vanishes on the mass shell,

$$\not{\partial}\psi = 0.$$

OK, but it might be useful to close the algebra even off shell.

* For those who wish to try: you need the Fierz identities (and some patience).

Supermultiplets

This can be done. Let us introduce an auxiliary scalar field F, with the lagrangian

$$\mathcal{L}_a = F^* F$$

(note the unusual dimension: m^2 instead of m). It is harmless: the equation of motion is $F = 0$. Now assume

$$\delta F = -i\,\bar{\eta}\,\gamma^\mu\,\partial_\mu \psi \qquad \delta F^* = i\,\partial_\mu \bar{\psi}\,\gamma^\mu\,\eta$$

and modify the fermion transformation rules as follows:

$$\delta\psi = P_L\left[-i\partial_\mu \phi\,\gamma^\mu \eta + F\eta\right] \qquad \delta\bar{\psi} = \left[i\bar{\eta}\,\gamma^\mu \partial_\mu \phi^* + \bar{\eta} F^*\right] P_R$$

(which amounts to nothing, because the auxiliary fields vanish on shell).

Supermultiplets

Things start getting complicated:

1. **We can no longer assume that $P_R\eta = \eta$, we must promote it to a full four-component spinor. In order not to increase the number of independent parameters, we assume it is a Majorana spinor:**

$$\eta = \begin{pmatrix} \eta \\ -\epsilon\eta^* \end{pmatrix}$$

2. **We now have to write explicitly the projector P_L in front of $\delta\psi$, in order to keep ψ left-handed.**

Nevertheless, let us press on ...

Supermultiplets

... It is immediate to check that

$$\mathcal{L}_{\text{chiral}} = \mathcal{L}_s + \mathcal{L}_f + \mathcal{L}_a$$

is our first supersymmetric lagrangian (not very exciting: it's a non-interacting theory), invariant under the transformation

$$\delta\phi = \bar{\eta}\,\psi \qquad \delta\phi^* = \bar{\psi}\,\eta$$

$$\delta\psi = P_L\left[-i\partial_\mu\phi\,\gamma^\mu\eta + F\eta\right] \qquad \delta\bar{\psi} = \left[i\bar{\eta}\,\gamma^\mu\partial_\mu\phi^* + \bar{\eta}F^*\right]P_R$$

$$\delta F = -i\,\bar{\eta}\,\gamma^\mu\,\partial_\mu\psi \qquad \delta F^* = i\,\partial_\mu\bar{\psi}\,\gamma^\mu\,\eta$$

with

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})X = -(\bar{\eta}_2\,\gamma^\mu P_R\,\eta_1 - \bar{\eta}_1\,\gamma^\mu P_R\,\eta_2)\,i\partial_\mu X$$

for $X = \phi, \psi, F$, both on and off the mass shell. This structure can be replicated for an arbitrary number of chiral supermultiplets.

Supermultiplets

The same results they take a much simpler form when recast in Weyl language. Define Weyl spinors through

$$\psi \to \begin{pmatrix} \psi \\ 0 \end{pmatrix} \qquad \eta \to \begin{pmatrix} \eta \\ -\epsilon\eta^* \end{pmatrix}$$

Recalling the definitions of γ matrices, we get

$$\delta\phi = \eta\,\psi$$

$$\delta\psi = -i(\sigma^\mu \epsilon\eta^*)\,\partial_\mu\phi + F\eta$$

$$\delta F = -i\,\eta^\dagger\,\bar\sigma^\mu\,\partial_\mu\psi$$

and

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})X = -(\eta_1^\dagger\,\bar\sigma^\mu\,\eta_2 - \eta_2^\dagger\,\bar\sigma^\mu\,\eta_1)\,i\partial_\mu X$$

Supermultiplets

Why do we need auxiliary fields? Count fermionic and bosonic degrees of freedom:

- **on shell: ψ has two helicity states (it is a Weyl fermion): $n_f = 2$**
  ϕ is a complex scalar: $n_b = 2$
- **off shell: ψ is a complex two-component spinor: $n_f = 4$**
  ϕ is a complex scalar: $n_b = 2$
  F is a complex scalar: $n_b = 2$

Without the auxiliary fields, the counting rule $n_f = n_b$ would be violated off the mass shell.

Ordinary symmetries: a reminder

Consider a lagrangian density

$$\mathcal{L}(\phi, \partial\phi)$$

functions of a set of fields $\phi_i, i = 1, \ldots, n$. The transformation

$$\delta\phi_i = i\eta^A T^A_{ij}\phi_j; \quad (T^A)^\dagger = T^A; \quad [T^A, T^B] = i f^{ABC} T^C$$

is a symmetry transformation if

$$\delta\mathcal{L} = \eta^A \partial^\mu K^A_\mu \;\Rightarrow\; \delta S[\phi] = \delta \int d^4x\, \mathcal{L} = 0$$

As a consequence, a set of *conserved currents* exists:

$$\partial^\mu J^A_\mu = 0; \quad J^A_\mu = T^A_{ij}\phi_j \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi_i} - K^A_\mu$$

$$Q^A(t) = \int d^3x\, J^A_0(t, \mathbf{x}); \quad \frac{d}{dt}Q^A(t) = 0; \quad [Q^A, Q^B] = i f^{ABC} Q^C$$

Superalgebra

The same procedure can be applied to our supersymmetry: a straightforward task.

The supersymmetry charge Q (a Weyl spinor!) is the space integral of the 0 component of a conserved current J_μ.

One finds that its components obey the anticommutator algebra

$$\{Q, Q^\dagger\} = 2\sigma^\mu P_\mu; \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0$$

as anticipated.

Superalgebra

Field representations

- Field theories, describing the interactions of fundamental particles, are invariant under Lorentz transformations
- Lorentz Group is equivalent to $SU(2)_L \times SU(2)_R$, and therefore Lorentz group representations are labelled by two indeces (m,n)
- Spin 0-scalars have (0,0)
- Spin 1/2 fermions (1/2,0) (left-handed) and (0,1/2) (right-handed)
- Spin 1-vectors (1/2,1/2)

SUSY Irreducible representations

Additional to J_{mn} and P_m spinor generators

$$Q_\alpha^r \quad , \quad \bar{Q}_{\dot{\beta},s} \qquad r,s = 1,2,\ldots,N$$

The N –extended superalgebra is

$$\{Q_\alpha^r, \bar{Q}_{\dot{\beta},s}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta_r^s$$

$$\{Q_\alpha^r, Q_\beta^s\} = \{\bar{Q}_{\dot{\alpha},r}, \bar{Q}_{\dot{\beta},s}\} = 0$$

$$[P_m, Q_\alpha^r] = [P_m, \bar{Q}_{\dot{\beta},s}] = 0$$

$$[P_m, P_n] = 0$$

Irreducible representations

From $[P_m, Q_\alpha^r] = [P_m, \bar{Q}_{\dot{\beta},s}] = 0$ it follows P^2 is a Casimir Operator

➔ $P^2 = M^2$ all members of the multiplet have equal masses

Massive Irreducible Representation

Let us consider representation with $P^2 = -M^2$

In the rest frame we have $P_m = (-M, 0, 0, 0)$

$$\{Q_\alpha^r, \bar{Q}_{\dot{\beta},s}\} = 2M\delta_{\alpha\dot{\beta}}\delta_r^s$$

$$\{Q_\alpha^r, Q_\beta^s\} = \{\bar{Q}_{\dot{\alpha},r}, \bar{Q}_{\dot{\beta},s}\} = 0$$

Irreducible representations

Let us introduce new operators

$$a_\alpha^r = \frac{1}{\sqrt{2M}} Q_\alpha^r \qquad \text{and} \qquad (a_\alpha^r)^+ = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha},r}$$

They are creation and annihilation operators satisfying

$$\{a_\alpha^r, (a_\beta^s)^+\} = 2M\delta_{\alpha\dot{\beta}}\delta_s^r$$

$$\{a_\alpha^r, a_\beta^s\} = \{(a_\alpha^r)^+, (a_\beta^s)^+\} = 0$$

Let us define Cliford vacuum Ω

$$a_\alpha^r \Omega = 0 \qquad P^2\Omega = -M^2\Omega$$

Irreducible representations

where Ω transforms under some irreducible representation of the Poinkare group. Then the states

$$\Omega^{(n)\alpha_1,\ldots,\alpha_n}_{r_1,\ldots,r_n} = \frac{1}{\sqrt{n}}(a^{r_1}_{\alpha_1})^+ \ldots (a^{r_n}_{\alpha_n})^+ \Omega$$

form irreducible representation of the N – extended supergroup

Due to anticommutation of the creation operators the supermultiplet has finite dimension

For every n we have $\binom{2N}{n}$ states and $d = \sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N}$

As a result we have the same number of bosonic and fermionic degrees of freedom

$$N_f = 2^{2N-1} \text{ and } N_B = 2^{2N-1}$$

The maximal spin in the representation is $s = \frac{1}{2}N$

Irreducible representations

Example N=1

s=0 s=1/2 s=0

$$\Omega \qquad (a_\alpha)^+\Omega, \qquad \frac{1}{\sqrt{2}}(a_\alpha)^+(a_\beta)^+\Omega = \frac{1}{2\sqrt{2}}\varepsilon^{\alpha\beta}(a^\gamma)^+(a_\gamma)^+\Omega$$

If Ω_j transforms under irreducible representation with spin j

Then the supermultiplet includes states with spins (j, j+1/2, j-1/2, j)

For N=1 and N=2 we have the following multiplcities of the IR

N=1

spin	Ω_0	$\Omega_{1/2}$	Ω_1	$\Omega_{3/2}$
0	2	1		
1/2	1	2	1	
1		1	2	1
3/2			1	2
2				1

N=2

spin	Ω_0	$\Omega_{1/2}$	Ω_1
0	5	4	1
1/2	4	6	4
1	1	4	6
3/2		1	4
2			1

Superspace

Superspace

- In order to describe supersymmetric theories, it proves convenient to introduce the concept of superspace.
- Apart from the ordinary coordinates x^μ, one introduces new anticommuting spinor coordinates θ^α and $\bar{\theta}_{\dot{\alpha}}$; $[\theta] = [\bar{\theta}] =$ -1/2.
- One can also define derivatives

$$\{\theta_\alpha, \theta_\beta\} = 0; \qquad \theta\theta\theta = 0; \qquad [\theta Q, \bar{\theta}\bar{Q}] = 2\theta\bar{\theta}\sigma^\mu P_\mu$$

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}; \qquad \partial_\alpha\theta^\beta = \delta_\alpha^\beta; \qquad \partial_\alpha(\theta^\beta\theta_\beta) = 2\theta_\alpha \tag{14}$$

Supersymmetry representation

- Supersymmetry is a particular translation in superspace, characterized by a Grassman parameter ξ.
- Supersymmetry generators may be given as derivative operators

$$Q_\alpha = i\left[-\partial_\theta - i\sigma^\mu\bar{\theta}\partial_\mu\right] \tag{15}$$

- Superspace allows to represent fermion and boson fields by the same superfield, by fields in superspace
- The operator

$$\bar{D} = -\partial_{\dot{\alpha}} + i\theta\sigma^\mu\partial_\mu$$

commutes with the supersymmetry transformations.

- So, if a field depends only on the variable $y^\mu = x^\mu - i\bar{\theta}\sigma^\mu\theta$, the supersymmetric transformation of it depends also on the y.

Chiral Fields

- A generic scalar, chiral field is given by

$$\begin{aligned}\Phi(x,\theta,\bar\theta=0) &= A(x)+\sqrt{2}\,\theta\,\psi(x)+\theta^2F(x)\\ \Phi(x,\theta,\bar\theta) &= \exp(-i\partial_\mu\theta\sigma^\mu\bar\theta)\,\Phi(x,\theta,\bar\theta=0)\end{aligned} \qquad (16)$$

- A, ψ and F are the scalar, fermion and auxiliary components.
- Under supersymmetric transformations, the components of chiral fields transform like

$$\begin{aligned}\delta A &= \sqrt{2}\xi\psi, \qquad\qquad \delta F=-i\sqrt{2}\bar\xi\bar\sigma^\mu\partial_\mu\psi\\ \delta\psi &= -i\sqrt{2}\sigma^\mu\bar\xi\partial_\mu A+\sqrt{2}\xi F\end{aligned} \qquad (17)$$

- The F component transforms like a total derivative.

Expansion of Chiral Superfield

- In the above, we have only used the form of the chiral field at $\bar{\theta} = 0$.
- However, for many applications, the full expression of the chiral superfield is necessary. It is given by

$$\Phi(x, \theta, \bar{\theta}) = A(x) + i\partial^\mu A(x)\theta\sigma_\mu\bar{\theta} - \frac{1}{4}\partial^2 A(x)\theta^2\bar{\theta}^2$$
$$+\theta\psi(x) + i\frac{\theta^2}{2}\partial^\mu\psi(x)\sigma_\mu\bar{\theta} + F(x)\theta^2 \quad (20)$$

Vector Superfields

- Vector Superfields are generic hermitian fields. The minimal irreducible representations may be obtained by

$$V(x,\theta,\bar\theta) = -\left(\theta\sigma^\mu\bar\theta\right)V_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D \qquad (21)$$

- Vector Superfields contain a regular vector field V_μ, its fermionic supersymmetric partner λ and an auxiliary scalar field D.
- Looking at the form of Q_α, it is easy to see that the D-component of a vector field transform like a total derivative.
- $D = [V] + 2$; $[V_\mu] = [V] + 1$; $[\lambda] = [V] + 3/2$. If V_μ describes a physical gauge field, then [V] = 0.

Building SUSY SM

Masses and non-gauge interactions

It can be shown that the most general renormalizable supersymmetric term has the form (in Weyl notation)

$$\mathcal{L}_{\rm ng} = -\frac{1}{2} W_{ij}\, \psi_i \psi_j + W_i\, F_i + \text{h.c.}$$

where

$$W_i = \frac{\partial W}{\partial \phi_i} \qquad W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}$$

and W is the superpotential:

$$W = \frac{1}{2} M_{ij}\, \phi_i \phi_j + \frac{1}{6} y_{ijk}\, \phi_i \phi_j \phi_k,$$

The constants M_{ij} and y_{ijk} must be totally symmetric in their indices.

The equations of motion for the auxiliary fields F_i are now

$$F_i^* = -W_i = M_{ij}\,\phi_j + \frac{1}{2} y_{ijk}\,\phi_j \phi_k$$

and therefore

$$\begin{aligned} \mathcal{L}_{\rm ng} + \mathcal{L}_a &= -\frac{1}{2}\,W_{ij}\,\psi_i\psi_j - \frac{1}{2}\,W^*_{ij}\,\psi_i^\dagger\psi_j^\dagger + W_i F_i + W_i^* F_i^* - F_i\,F_i^* \\ &= -\frac{1}{2}\,W_{ij}\,\psi_i\psi_j - \frac{1}{2}\,W^*_{ij}\,\psi_i^\dagger\psi_j^\dagger - W_i\,W_i^* \end{aligned}$$

The scalar potential is entirely determined by the superpotential:

$$\begin{aligned} V(\phi) &= W_i\,W_i^* = F_i\,F_i^* \\ &= M_{ij} M^*_{il}\,\phi_j\phi_l^* \qquad \text{scalar masses} \\ &\quad + \frac{1}{2}\,M_{ij}\,y^*_{ilm}\,\phi_j\phi_l^*\phi_m^* + \frac{1}{2}\,M^*_{il}\,y_{ijk}\,\phi_j\phi_k\phi_l^* \qquad \text{trilinear couplings} \\ &\quad + \frac{1}{4}\,y_{ijk}\,y^*_{ilm}\,\phi_j\phi_k\phi_l^*\phi_m^* \qquad \text{quartic couplings} \end{aligned}$$

The remaining terms contain mass terms for the fermions and Yukawa couplings:

$$
\begin{aligned}
-\frac{1}{2} W_{ij}\, \psi_i \psi_j - \frac{1}{2} W^*_{ij}\, \psi^\dagger_i \psi^\dagger_j \quad = \quad & -\frac{1}{2} M_{ij}\, \psi_i \psi_j - \frac{1}{2} M^*_{ij}\, \psi^\dagger_i \psi^\dagger_j \\
& -\frac{1}{2} y_{ijk}\, \phi_i \psi_j \psi_k - \frac{1}{2} y^*_{ijk}\, \phi^*_i \psi^\dagger_j \psi^\dagger_k
\end{aligned}
$$

Many interaction terms, but couplings related by supersymmetry.

Some comments:

- The scalar potential is positive semi-definite: (almost) no problems with the stability of the ground state.
- The quartic scalar self-coupling and the fermion-fermion-scalar Yukawa coupling are related: $\lambda_S = |\lambda_f|^2$.
- It is easy to check that scalars and fermions have the same squared mass matrix

$$(M^2)_{ij} = M_{ik} M^*_{kj}$$

  which is symmetric and can be diagonalized.
- The analiticity of W requires the introduction of two Higgs doublets in the supersymmetric version of the Standard Model.

In summary, we have the following

Recipe

for non-gauge interaction terms of a set of chiral supermultiplets ϕ_i, ψ_i, F_i:

1. Give a superpotential

$$W = \frac{1}{2} M_{ij}\, \phi_i \phi_j + \frac{1}{6} y_{ijk}\, \phi_i \phi_j \phi_k,$$

   with M_{ij} and y_{ijk} completely symmetric.

2. Build the interaction lagrangian as

$$\mathcal{L}_{\rm ng} = -\frac{1}{2} W_{ij}\, \psi_i \psi_j + W_i\, F_i + \text{h.c.}$$

   where

$$W_i = \frac{\partial W}{\partial \phi_i} \qquad\qquad W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}$$

Gauge interactions

A gauge supermultiplet contains a vector field A_μ^a and a Weyl fermion λ^a. An infinitesimal gauge symmetry transformation with parameter α^a acts on λ^a as

$$\delta_{\text{gauge}} \lambda^a = g\, f^{abc} \lambda^b \alpha^c$$

$a = 1, \ldots, 8$, $f^{abc} \to f^{abc}_{SU(3)}$ and $g \to g_s$ for color
$a = 1, \ldots, 3$, $f^{abc} \to \epsilon^{abc}$ and $g \to g$ for weak isospin
$a = 1$, $f^{abc} \to 0$ and $g \to g'$ for weak hypercharge.

Counting fermionic and bosonic degrees of freedom, we see that we should add one real scalar auxiliary field, D^a (off shell, $n_b = 3$ from A_μ and $n_f = 4$ from λ). Then

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\lambda^{a\dagger} \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a$$

where

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c$$

The corresponding supersymmetry transformation laws are easily worked out, requiring that δA^a_μ be real, and δD^a proportional to the equation of motion of λ^a:

$$\delta A^a_\mu = -\frac{1}{\sqrt{2}}\left(\eta^\dagger\,\bar\sigma_\mu\,\lambda^a + \lambda^{a\dagger}\,\bar\sigma_\mu\,\eta\right)$$

$$\delta\lambda^a = -\frac{i}{2\sqrt{2}}\,\sigma^\mu\bar\sigma^\nu\eta\,F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\,D^a\eta$$

$$\delta D^a = -\frac{i}{\sqrt{2}}\left[\eta^\dagger\,\bar\sigma^\mu D_\mu\,\lambda^a - (D_\mu\lambda^a)^\dagger\,\bar\sigma^\mu\,\eta\right]$$

With these definitions,

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})X = -(\eta_1^\dagger\,\bar\sigma^\mu\,\eta_2 - \eta_2^\dagger\,\bar\sigma^\mu\,\eta_1)\,iD_\mu X$$

on $X = F^a_{\mu\nu}, \lambda^a, D^a$.

Back to the chiral supermultiplet $X = \phi, \psi, F$. All of them must be in the same representation of the gauge group:

$$\delta_{\text{gauge}} X_i = i\, g\alpha^a \,(T^a)_{ij}\, X_j$$

where T^a are the generators of the gauge group in the representation of X. Ordinary derivatives must be replaced by covariant derivatives everywhere:

$$\partial_\mu \to D_\mu = \partial_\mu + i\, g\, A_\mu^a\, T^a$$

Is this enough to turn our simple model into a gauge-invariant supersymmetric theory?

Not quite: there are other possible gauge invariant and renormalizable terms that can be formed out of the fields in the gauge and chiral supermultiplets:

$$\phi^* T^a \psi \lambda^a \qquad (\phi^* T^a \phi) D^a$$

They are not taken into account by the covariant derivatives, because they involve the superpartners of the gauge fields.

They can be added to our supersymmetric lagrangian, with appropriate coefficients, provided the transformation laws of the auxiliary field F is slightly modified:

$$\delta\phi = \eta\,\psi$$

$$\delta\psi = -i(\sigma^\mu \epsilon \eta^*)\, D_\mu \phi + F\eta$$

$$\delta F = -i\,\eta^\dagger\,\bar{\sigma}^\mu\, D_\mu \psi + g\sqrt{2}(T^a\phi)\lambda^{a\dagger}\eta^\dagger$$

Then

$$\mathcal{L} = \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{chiral}} + g\sqrt{2}\left(\phi^* T^a \psi \lambda^a + \phi T^a \lambda^{a\dagger} \psi^\dagger\right) + g(\phi^* T^a \phi) D^a$$

is supersymmetric, provided the superpotential is gauge-invariant:

$$\delta_{\mathrm{gauge}} W = W_i (T^a)_{ij} \phi_j = 0$$

The equations of motion for D^a are now

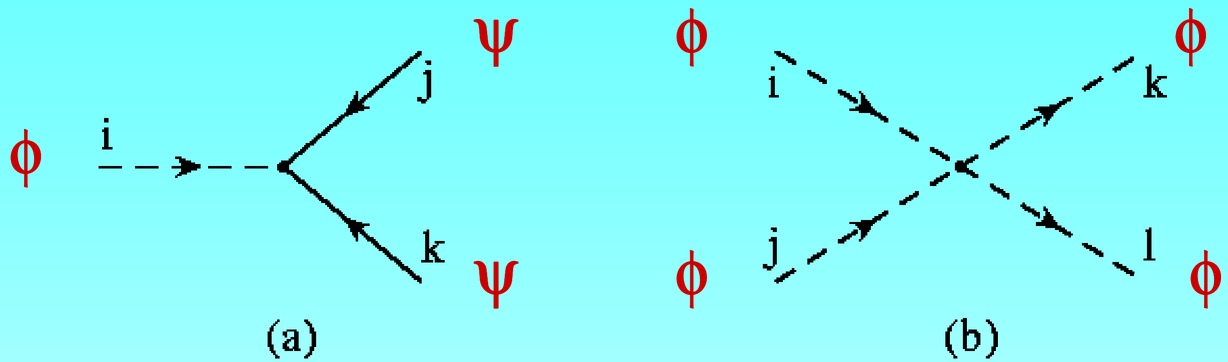
$$D^a = -g(\phi^* T^a \phi)$$

and the scalar potential becomes

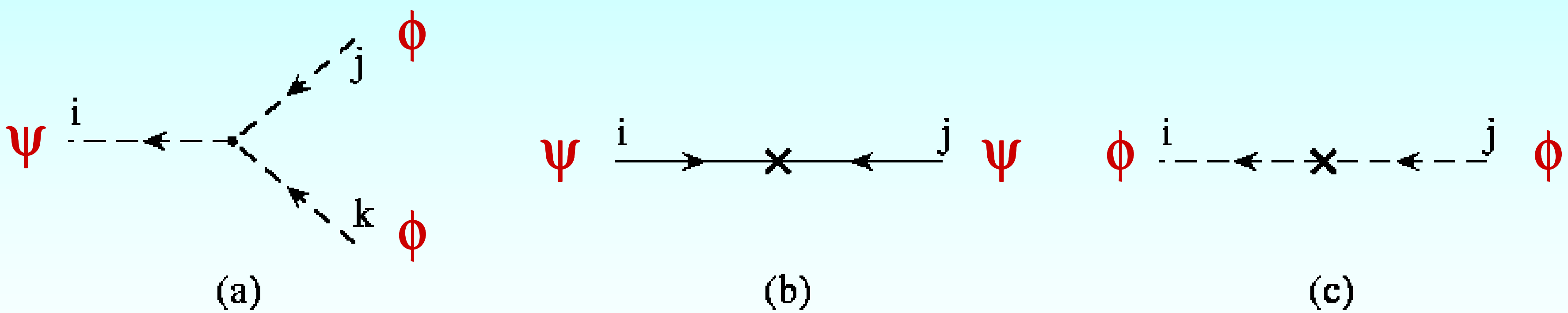
$$V(\phi) = F_i^* F_i + \frac{1}{2} D^a D^a = W_i^* W_i + \frac{1}{2} g^2 (\phi^* T^a \phi)(\phi^* T^a \phi)$$

completely determined by the auxiliary fields. This fact is relevant for the discussion of spontaneous supersymmetry breaking.

Couplings in a supersymmetric gauge theory

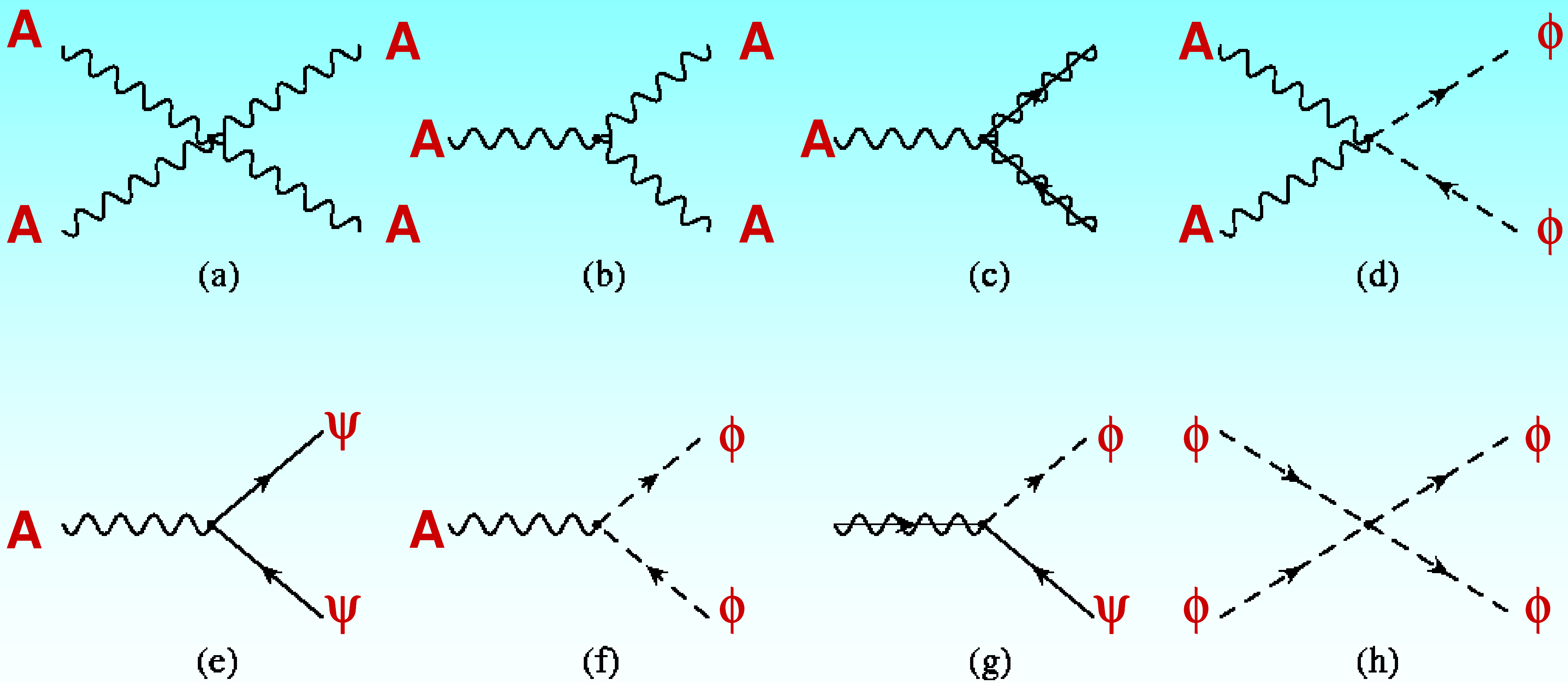


y_{ijk} $\quad$ $y_{ijm}y^*_{klm}$



$M^*_{im}y_{jkm}$ $\quad$ M_{ij} $\quad$ $M^*_{ik}M_{kj}$

Gauge couplings



Soft supersymmetry breaking

The most general soft supersymmetry breaking lagrangian is given by

$$\mathcal{L}_{\rm soft} = -(m^2)_{ij}\phi_i\phi_j^* - \left(\frac{1}{2}\,m_\lambda\,\lambda^a\lambda^a + \frac{1}{2}\,b_{ij}\phi_i\phi_j + \frac{1}{6}\,a_{ijk}\phi_i\phi_j\phi_k + \mathrm{h.c.}\right)$$

It can be shown that a theory with exact supersymmetry, plus $\mathcal{L}_{\rm soft}$, is free of quadratic divergences.

In principle, there could also be

$$-\frac{1}{2}\,c_{ijk}\phi_i^*\phi_j\phi_k + \mathrm{h.c.}$$

Usually negligible in most supersymmetry breaking scenarios.

A word on gaugino masses. A mass term for a left-handed Weyl spinor

$$\xi^T \epsilon \xi + \mathrm{h.c.}$$

is called a **Majorana mass term**. It is allowed by Lorentz invariance, but generally forbidden if ξ belongs to a complex representation of a gauge group. A simple example: a phase transformation $\delta\xi = i\alpha\xi$. Matter fermions in the Standard Model (with the possible exception of a **right-handed neutrino**) cannot have a Majorana mass term.

The case of gauginos λ^a is different: they transform according to the adjoint representation,

$$\delta_{\mathrm{gauge}} \lambda_a = g\, f_{abc} \lambda_b \alpha_c$$

which is self-conjugate. This gives

$$\delta_{\mathrm{gauge}}(\lambda_a^T \epsilon \lambda_a) = g\, f_{abc} \alpha_c (\lambda_b^T \epsilon \lambda_a + \lambda_a^T \epsilon \lambda_b) = 0$$

because f_{abc} is completely antisymmetric.